Chapter 10

Potentials and Fields

Problem 10.1

$$\begin{aligned}
\Box^2 V + \frac{\partial L}{\partial t} &= \nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} + \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) + \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = \nabla^2 V + \frac{\partial}{\partial t}(\boldsymbol{\nabla}\cdot\mathbf{A}) = -\frac{1}{\epsilon_0}\rho. \checkmark \\
\Box^2\mathbf{A} - \boldsymbol{\nabla}L &= \nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \checkmark
\end{aligned}$$

Problem 10.2

(a) $W = \frac{1}{2}\int\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau$. At $t_1 = d/c$, $x \geq d = ct_1$, so $\mathbf{E} = 0$, $\mathbf{B} = 0$, and hence $\boxed{W(t_1) = 0.}$
At $T_2 = (d+h)/c$, $ct_2 = d+h$:

$$\mathbf{E} = -\frac{\mu_0\alpha}{2}(d+h-x)\,\hat{\mathbf{z}}, \quad \mathbf{B} = \frac{1}{c}\frac{\mu_0\alpha}{2}(d+h-x)\,\hat{\mathbf{y}},$$

so $B^2 = \frac{1}{c^2}E^2$, and

$$\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) = \epsilon_0\left(E^2 + \frac{1}{\mu_0\epsilon_0}\frac{1}{c^2}E^2\right) = 2\epsilon_0 E^2.$$

Therefore

$$W(t_2) = \frac{1}{2}(2\epsilon_0)\frac{\mu_0^2\alpha^2}{4}\int_d^{(d+h)}(d+h-x)^2\,dx\,(lw) = \frac{\epsilon_0\mu_0^2\alpha^2 lw}{4}\left[-\frac{(d+h-x)^3}{3}\right]_d^{d+h} = \boxed{\frac{\epsilon_0\mu_0^2\alpha^2 lwh^3}{12}.}$$

(b) $\mathbf{S}(x) = \frac{1}{\mu_0}(\mathbf{B}\times\mathbf{E}) = \frac{1}{\mu_0 c}E^2[-\hat{\mathbf{z}}\times(\pm\hat{\mathbf{y}})] = \pm\frac{1}{\mu_0 c}E^2\,\hat{\mathbf{x}} = \boxed{\pm\frac{\mu_0\alpha^2}{4c}(ct-|x|)^2\,\hat{\mathbf{x}}}$
(plus sign for $x > 0$, as here). For $|x| > ct$, $\mathbf{S} = 0$.
So the energy per unit time entering the box in this time interval is

$$\frac{dW}{dt} = P = \int \mathbf{S}(d)\cdot d\mathbf{a} = \boxed{\frac{\mu_0\alpha^2 lw}{4c}(ct-d)^2.}$$

Note that no energy flows out the top, since $\mathbf{S}(d+h) = 0$.

$$\text{(c)}\ W = \int_{t_1}^{t_2} P\,dt = \frac{\mu_0\alpha^2 lw}{4c}\int_{d/x}^{(d+h)/c} (ct-d)^2\,dt = \frac{\mu_0\alpha^2 lw}{4c}\left[\frac{(ct-d)^3}{3c}\right]_{d/c}^{(d+h)/c} = \boxed{\frac{\mu_0\alpha^2 lwh^3}{12c^2}.}$$

Since $1/c^2 = \mu_0\epsilon_0$, this agrees with the answer to (a).

Problem 10.3

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}.} \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = \boxed{0.}$$

This is a funny set of potentials for a $\boxed{\text{stationary point charge}}$ q at the origin. ($V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, $\mathbf{A} = 0$ would, of course, be the customary choice.) Evidently $\boxed{\rho = q\delta^3(\mathbf{r});\ \mathbf{J} = 0.}$

Problem 10.4

$$\begin{aligned}
\mathbf{E} &= -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = -A_0\cos(kx-\omega t)\,\hat{\mathbf{y}}(-\omega) = \boxed{A_0\omega\cos(kx-\omega t)\,\hat{\mathbf{y}},}\\
\mathbf{B} &= \boldsymbol{\nabla}\times\mathbf{A} = \hat{\mathbf{z}}\,\frac{\partial}{\partial x}\left[A_0\sin(kx-\omega t)\right] = \boxed{A_0k\cos(kx-\omega t)\,\hat{\mathbf{z}}.}
\end{aligned}$$

Hence $\boldsymbol{\nabla}\cdot\mathbf{E} = 0\ \checkmark$, $\boldsymbol{\nabla}\cdot\mathbf{B} = 0\ \checkmark$.

$$\boldsymbol{\nabla}\times\mathbf{E} = \hat{\mathbf{z}}\,\frac{\partial}{\partial x}\left[A_0\omega\cos(kx-\omega t)\right] = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}}, \qquad -\frac{\partial\mathbf{B}}{\partial t} = -A_0\omega k\sin(kx-\omega t)\,\hat{\mathbf{z}},$$

so $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\ \checkmark$.

$$\boldsymbol{\nabla}\times\mathbf{B} = -\hat{\mathbf{y}}\,\frac{\partial}{\partial x}\left[A_0k\cos(kx-\omega t)\right] = A_0k^2\sin(kx-\omega t)\,\hat{\mathbf{y}}, \qquad \frac{\partial\mathbf{E}}{\partial t} = A_0\omega^2\sin(kx-\omega t)\,\hat{\mathbf{y}}.$$

So $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$ *provided* $\boxed{k^2 = \mu_0\epsilon_0\omega^2,}$ or, since $c^2 = 1/\mu_0\epsilon_0$, $\boxed{\omega = ck.}$

Problem 10.5

$$V' = V - \frac{\partial\lambda}{\partial t} = 0 - \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r};}\ \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\lambda = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}} + \left(-\frac{1}{4\pi\epsilon_0}qt\right)\left(-\frac{1}{r^2}\,\hat{\mathbf{r}}\right) = \boxed{0.}$$

This gauge function transforms the "funny" potentials of Prob. 10.3 into the "ordinary" potentials of a stationary point charge.

Problem 10.6

Ex. 10.1: $\boldsymbol{\nabla}\cdot\mathbf{A} = 0;\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Both Coulomb and Lorentz.}}$

Prob. 10.3: $\boldsymbol{\nabla}\cdot\mathbf{A} = -\frac{qt}{4\pi\epsilon_0}\,\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = -\frac{qt}{\epsilon_0}\delta^3(\mathbf{r});\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Neither.}}$

Prob. 10.4: $\boldsymbol{\nabla}\cdot\mathbf{A} = 0;\ \frac{\partial V}{\partial t} = 0.$ $\boxed{\text{Both.}}$

Problem 10.7

Suppose $\nabla\cdot\mathbf{A} \neq -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. (Let $\nabla\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t} = \Phi$—some known function.) We want to pick λ such that $\mathbf{A}'$ and V' (Eq. 10.7) *do* obey $\nabla\cdot\mathbf{A}' = -\mu_0\epsilon_0\frac{\partial V'}{\partial t}$.

$$\nabla\cdot\mathbf{A}' + \mu_0\epsilon_0\frac{\partial V'}{\partial t} = \nabla\cdot\mathbf{A} + \nabla^2\lambda + \mu_0\epsilon_0\frac{\partial V}{\partial t} - \mu_0\epsilon_0\frac{\partial^2\lambda}{\partial t^2} = \Phi + \Box^2\lambda.$$

This will be zero provided we pick for λ the solution to $\Box^2\lambda = -\Phi$, which by hypothesis (and in fact) we know how to solve.

We *could* always find a gauge in which $V' = 0$, simply by picking $\lambda = \int_0^t V\,dt'$. We *cannot* in general pick $\mathbf{A} = 0$—this would make $\mathbf{B} = 0$. [Finding such a gauge function would amount to expressing $\mathbf{A}$ as $-\nabla\lambda$, and we know that vector functions *cannot* in general be written as gradients—only if they happen to have curl zero, which $\mathbf{A}$ (ordinarily) does *not*.]

Problem 10.8

¿From the product rule:

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla\frac{1}{\mathscr{r}}\right), \quad \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J}) + \mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right).$$

But $\nabla\frac{1}{\mathscr{r}} = -\nabla'\frac{1}{\mathscr{r}}$, since $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}'$. So

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) - \mathbf{J}\cdot\left(\nabla'\frac{1}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla\cdot\mathbf{J}) + \frac{1}{\mathscr{r}}(\nabla'\cdot\mathbf{J}) - \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right).$$

But

$$\nabla\cdot\mathbf{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = \frac{\partial J_x}{\partial t_r}\frac{\partial t_r}{\partial x} + \frac{\partial J_y}{\partial t_r}\frac{\partial t_r}{\partial y} + \frac{\partial J_z}{\partial t_r}\frac{\partial t_r}{\partial z},$$

and

$$\frac{\partial t_r}{\partial x} = -\frac{1}{c}\frac{\partial\mathscr{r}}{\partial x}, \quad \frac{\partial t_r}{\partial y} = -\frac{1}{c}\frac{\partial\mathscr{r}}{\partial y}, \quad \frac{\partial t_r}{\partial z} = -\frac{1}{c}\frac{\partial\mathscr{r}}{\partial z},$$

so

$$\nabla\cdot\mathbf{J} = -\frac{1}{c}\left[\frac{\partial J_x}{\partial t_r}\frac{\partial\mathscr{r}}{\partial x} + \frac{\partial J_y}{\partial t_r}\frac{\partial\mathscr{r}}{\partial y} + \frac{\partial J_z}{\partial t_r}\frac{\partial\mathscr{r}}{\partial z}\right] = -\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla\mathscr{r}).$$

Similarly,

$$\nabla'\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r}).$$

[The first term arises when we differentiate with respect to the *explicit* $\mathbf{r}'$, and use the continuity equation.] thus

$$\nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}\left[-\frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right] + \frac{1}{\mathscr{r}}\left[-\frac{\partial\rho}{\partial t} - \frac{1}{c}\frac{\partial\mathbf{J}}{\partial t_r}\cdot(\nabla'\mathscr{r})\right] - \nabla\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}}\frac{\partial\rho}{\partial t} - \nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)$$

(the other two terms cancel, since $\nabla\mathscr{r} = -\nabla'\mathscr{r}$). Therefore:

$$\nabla\cdot\mathbf{A} = \frac{\mu_0}{4\pi}\left[-\frac{\partial}{\partial t}\int\frac{\rho}{\mathscr{r}}\,d\tau - \int\nabla'\cdot\left(\frac{\mathbf{J}}{\mathscr{r}}\right)d\tau\right] = -\mu_0\epsilon_0\frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}\,d\tau\right] - \frac{\mu_0}{4\pi}\oint\frac{\mathbf{J}}{\mathscr{r}}\cdot d\mathbf{a}.$$

The last term is over the suface at "infinity", where $\mathbf{J} = 0$, so it's zero. Therefore $\nabla\cdot\mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$. ✓

Problem 10.9

(a) As in Ex. 10.2, for $t < r/c$, $\mathbf{A} = 0$; for $t > r/c$,

$$\begin{aligned}
\mathbf{A}(r,t) &= \left(\frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\right) 2 \int\limits_0^{\sqrt{(ct)^2-r^2}} \frac{k(t-\sqrt{r^2+z^2}/c)}{\sqrt{r^2+z^2}}\,dz = \frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}} \left\{ t \int\limits_0^{\sqrt{(ct)^2-r^2}} \frac{dz}{\sqrt{r^2+z^2}} - \frac{1}{c} \int\limits_0^{\sqrt{(ct)^2-r^2}} dz \right\} \\
&= \left(\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}}\right) \left[t \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) - \frac{1}{c}\sqrt{(ct)^2-r^2} \right]. \quad \text{Accordingly,}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}(r,t) &= -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}} \left\{ \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \right. \\
&\qquad \left. t\left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right)\left(\frac{1}{r}\right)\left(c+\frac{1}{2}\frac{2c^2t}{\sqrt{(ct)^2-r^2}}\right) - \frac{1}{2c}\frac{2c^2t}{\sqrt{(ct)^2-r^2}} \right\} \\
&= -\frac{\mu_0 k}{2\pi}\,\hat{\mathbf{z}} \left\{ \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right) + \frac{ct}{\sqrt{(ct)^2-r^2}} - \frac{ct}{\sqrt{(ct)^2-r^2}} \right\} \\
&= \boxed{-\frac{\mu_0 k}{2\pi} \ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)\hat{\mathbf{z}}} \qquad \text{(or zero, for } t < r/c\text{).}
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}(r,t) &= -\frac{\partial A_z}{\partial r}\,\hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 k}{2\pi} \left\{ t \left(\frac{r}{ct+\sqrt{(ct)^2-r^2}}\right) \frac{\left[r\frac{1}{2}\frac{(-2r)}{\sqrt{(ct)^2-r^2}} - ct - \sqrt{(ct)^2-r^2}\right]}{r^2} - \frac{1}{2c}\frac{(-2r)}{\sqrt{(ct)^2-r^2}} \right\} \hat{\boldsymbol{\phi}} \\
&= -\frac{\mu_0 k}{2\pi} \left\{ \frac{-ct^2}{r\sqrt{(ct)^2-r^2}} + \frac{r}{c\sqrt{(ct)^2-r^2}} \right\} \hat{\boldsymbol{\phi}} = -\frac{\mu_0 k}{2\pi}\frac{(-c^2t^2+r^2)}{rc\sqrt{(ct)^2-r^2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{\mu_0 k}{2\pi r c}\sqrt{(ct)^2-r^2}\,\hat{\boldsymbol{\phi}}.}
\end{aligned}$$

(b) $\mathbf{A}(r,t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}} \int_{-\infty}^{\infty} \frac{q_0\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz$. But $\mathscr{r} = \sqrt{r^2+z^2}$, so the integrand is even in z:

$$\mathbf{A}(r,t) = \left(\frac{\mu_0 q_0}{4\pi}\,\hat{\mathbf{z}}\right) 2 \int_0^{\infty} \frac{\delta(t-\mathscr{r}/c)}{\mathscr{r}}\,dz.$$

Now $z = \sqrt{\mathscr{r}^2 - r^2} \Rightarrow dz = \frac{1}{2}\frac{2\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}} = \frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}}$, and $z = 0 \Rightarrow \mathscr{r} = r$, $z = \infty \Rightarrow \mathscr{r} = \infty$. So:

$$\mathbf{A}(r,t) = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}} \int_r^{\infty} \frac{1}{\mathscr{r}}\delta\left(t-\frac{\mathscr{r}}{c}\right)\frac{\mathscr{r}\,d\mathscr{r}}{\sqrt{\mathscr{r}^2-r^2}}.$$

Now $\delta(t-\mathscr{r}/c) = c\delta(\mathscr{r}-ct)$ (Ex. 1.15); therefore $\mathbf{A} = \frac{\mu_0 q_0}{2\pi}\,\hat{\mathbf{z}}\,c\int_r^\infty \frac{\delta(\mathscr{r}-ct)}{\sqrt{\mathscr{r}^2-r^2}}\,d\mathscr{r}$, so

$$\begin{aligned}
\mathbf{A}(r,t) &= \frac{\mu_0 q_0 c}{2\pi}\frac{1}{\sqrt{(ct)^2-r^2}}\,\hat{\mathbf{z}} \quad \text{(or zero, if } ct<r\text{);}\\
\mathbf{E}(r,t) &= -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{2c^2 t}{[(ct)^2-r^2]^{3/2}}\,\hat{\mathbf{z}} = \boxed{\frac{\mu_0 q_0 c^3 t}{2\pi[(ct)^2-r^2]^{3/2}}\,\hat{\mathbf{z}}} \quad \text{(or zero, for } t<r/c\text{);}\\
\mathbf{B}(r,t) &= -\frac{\partial \mathbf{A}_z}{\partial t}\,\hat{\boldsymbol{\phi}} = -\frac{\mu_0 q_0 c}{2\pi}\left(-\frac{1}{2}\right)\frac{-2r}{[(ct)^2-r^2]^{3/2}}\,\hat{\boldsymbol{\phi}} = \boxed{\frac{-\mu_0 q_0 c r}{2\pi[(ct)^2-r^2]^{3/2}}\,\hat{\boldsymbol{\phi}}} \quad \text{(or zero, for } t<r/c\text{).}
\end{aligned}$$

Problem 10.10

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}(t_r)}{\mathscr{r}}\,dl = \frac{\mu_0 k}{4\pi}\int\frac{(t-\mathscr{r}/c)}{\mathscr{r}}\,d\mathbf{l} = \frac{\mu_0 k}{4\pi}\left\{t\int\frac{d\mathbf{l}}{\mathscr{r}} - \frac{1}{c}\int d\mathbf{l}\right\}.$$

But for the complete loop, $\int d\mathbf{l} = 0$, so $\mathbf{A} = \frac{\mu_0 k t}{4\pi}\left\{\frac{1}{a}\int_1 d\mathbf{l} + \frac{1}{b}\int_2 d\mathbf{l} + 2\,\hat{\mathbf{x}}\int_a^b \frac{dx}{x}\right\}$. Here $\int_1 d\mathbf{l} = 2a\,\hat{\mathbf{x}}$ (inner circle), $\int_2 d\mathbf{l} = -2b\,\hat{\mathbf{x}}$ (outer circle), so

$$\mathbf{A} = \frac{\mu_0 k t}{4\pi}\left[\frac{1}{a}(2a) + \frac{1}{b}(-2b) + 2\ln(b/a)\right]\hat{\mathbf{x}} \Rightarrow \boxed{\mathbf{A} = \frac{\mu_0 k t}{2\pi}\ln(b/a)\,\hat{\mathbf{x}},} \quad \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \boxed{-\frac{\mu_0 k}{2\pi}\ln(b/a)\,\hat{\mathbf{x}}.}$$

The changing magnetic field induces the electric field. Since we only know $\mathbf{A}$ at *one point* (the center), we can't compute $\nabla\times\mathbf{A}$ to get $\mathbf{B}$.

Problem 10.11

In this case $\dot{\rho}(\mathbf{r},t) = \dot{\rho}(\mathbf{r},0)$ and $\dot{\mathbf{J}}(\mathbf{r},t) = 0$, so Eq. 10.29 ⇒

$$\begin{aligned}
\mathbf{E}(\mathbf{r},t) &= \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',0)+\dot{\rho}(\mathbf{r}',0)t_r}{\mathscr{r}^2} + \frac{\dot{\rho}(\mathbf{r}',0)}{c\mathscr{r}}\right]\hat{\mathscr{r}}\,d\tau', \text{ but } t_r = t-\frac{\mathscr{r}}{c} \text{ (Eq. 10.18), so}\\
&= \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\mathbf{r}',0)+\dot{\rho}(\mathbf{r}',0)t}{\mathscr{r}^2} - \frac{\dot{\rho}(\mathbf{r}',0)(\mathscr{r}/c)}{\mathscr{r}^2} + \frac{\dot{\rho}(\mathbf{r}',0)}{c\mathscr{r}}\right]\hat{\mathscr{r}}\,d\tau' = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{r}',t)}{\mathscr{r}^2}\,\hat{\mathscr{r}}\,d\tau'. \quad \text{qed}
\end{aligned}$$

Problem 10.12

In this approximation we're dropping the higher derivatives of $\mathbf{J}$, so $\dot{\mathbf{J}}(t_r) = \dot{\mathbf{J}}(t)$, and Eq. 10.31 ⇒

$$\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int\frac{1}{\mathscr{r}^2}\left[\mathbf{J}(\mathbf{r}',t) + (t_r-t)\dot{\mathbf{J}}(\mathbf{r}',t) + \frac{\mathscr{r}}{c}\dot{\mathbf{J}}(\mathbf{r}',t)\right]\times\hat{\mathscr{r}}\,d\tau', \text{ but } t_r - t = -\frac{\mathscr{r}}{c} \text{ (Eq. 10.18), so}\\
&= \frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{r}',t)\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'. \quad \text{qed}
\end{aligned}$$

Problem 10.13

At time t the charge is at $\mathbf{r}(t) = a[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$, so $\mathbf{v}(t) = \omega a[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$. Therefore $\boldsymbol{\mathscr{r}} = z\,\hat{\mathbf{z}} - a[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}]$, and hence $\mathscr{r}^2 = z^2 + a^2$ (of course), and $\mathscr{r} = \sqrt{z^2+a^2}$.

$$\hat{\mathscr{r}}\cdot\mathbf{v} = \frac{1}{\mathscr{r}}(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = \frac{1}{\mathscr{r}}\left\{-\omega a^2[-\sin(\omega t_r)\cos(\omega t_r) + \sin(\omega t_r)\cos(\omega t_r)]\right\} = 0, \text{ so } \left(1-\frac{\hat{\mathscr{r}}\cdot\mathbf{v}}{c}\right) = 1.$$

Therefore

$$V(z,t) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{z^2+a^2}}};\ \mathbf{A}(z,t) = \boxed{\frac{q\omega a}{4\pi\epsilon_0 c^2\sqrt{z^2+a^2}}[-\sin(\omega t_r)\,\hat{\mathbf{x}} + \cos(\omega t_r)\,\hat{\mathbf{y}}),}\ \text{where } \boxed{t_r = t - \frac{\sqrt{z^2+a^2}}{c}.}$$

Problem 10.14

Term under square root in (Eq. 9.98) is:

$$\begin{aligned} I &= c^4t^2 - 2c^2t(\mathbf{r}\cdot\mathbf{v}) + (\mathbf{r}\cdot\mathbf{v})^2 + c^2r^2 - c^4t^2 - v^2r^2 + v^2c^2t^2 \\ &= (\mathbf{r}\cdot\mathbf{v})^2 + (c^2-v^2)r^2 + c^2(vt)^2 - 2c^2(\mathbf{r}\cdot\mathbf{v}t). \quad \text{put in } \mathbf{v}t = \mathbf{r} - \mathbf{R}^2. \\ &= (\mathbf{r}\cdot\mathbf{v})^2 + (c^2-v^2)r^2 + c^2(r^2+R^2-2\mathbf{r}\cdot\mathbf{R}) - 2c^2(r^2 - \mathbf{r}\cdot\mathbf{R}) = (\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2 + c^2R^2. \end{aligned}$$

but

$$\begin{aligned} (\mathbf{r}\cdot\mathbf{v})^2 - r^2v^2 &= ((\mathbf{R}+\mathbf{v}t)\cdot\mathbf{v})^2 - (\mathbf{R}+\mathbf{v}t)^2v^2 \\ &= (\mathbf{R}\cdot\mathbf{v})^2 + v^4t^2 + 2(\mathbf{R}\cdot\mathbf{v})v^2t - R^2v^2 - 2(\mathbf{R}\cdot\mathbf{v})tv^2 - v^2t^2v^2 \\ &= (\mathbf{R}\cdot\mathbf{v})^2 - R^2v^2 = R^2v^2\cos^2\theta - R^2v^2 = -R^2v^2\left(1-\cos^2\theta\right) \\ &= -R^2v^2\sin^2\theta. \end{aligned}$$

Therefore

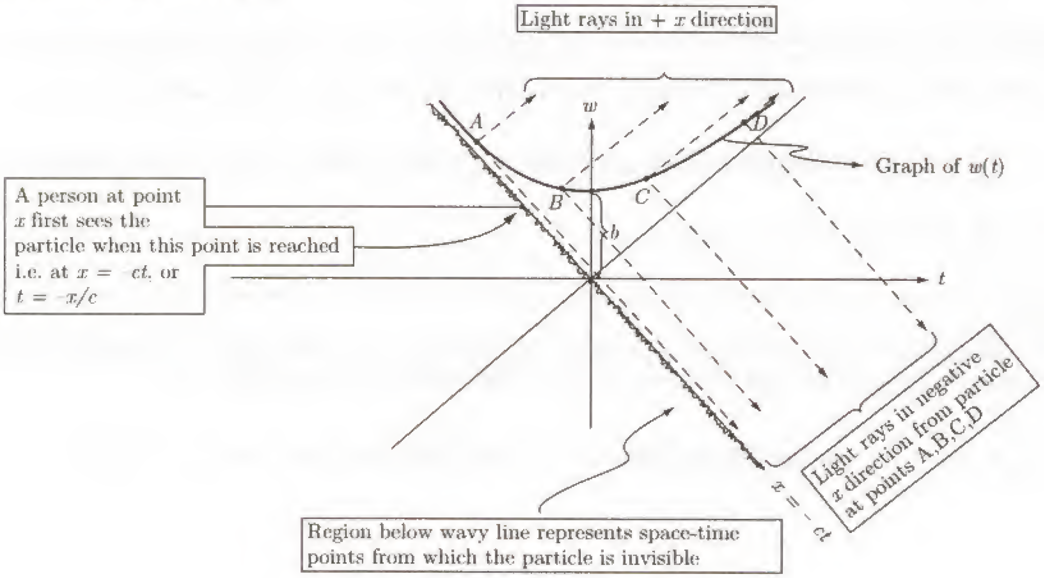
$$I = -R^2v^2\sin^2\theta + c^2R^2 = c^2R^2\left(1 - \frac{v^2}{c^2}\sin^2\theta\right).$$

Hence

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-\frac{v^2}{c^2}\sin^2\theta}}. \quad \text{qed}$$

Problem 10.15

Once seen, from a given point x, the particle will forever remain in view—to disappear it would have to travel faster than light.


Light rays in $+x$ direction
w
A
B
C
D
b
Graph of $w(t)$
t
A person at point x first sees the particle when this point is reached i.e. at $x = -ct$, or $t = -x/c$
$w = -t$
Light rays in negative x direction from particle at points A,B,C,D
Region below wavy line represents space-time points from which the particle is invisible

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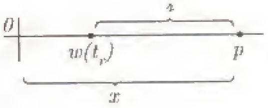
Problem 10.16

First calculate t_r: $t_r = t - |\mathbf{r} - \mathbf{w}(t_r)|/c \Rightarrow$



$$-c(t_r - t) = x - \sqrt{b^2 + c^2t_r^2} \Rightarrow c(t_r - t) + x = \sqrt{b^2 + c^2t_r^2};$$

$$c^2t_r^2 - 2c^2t_rt + c^2t^2 + 2xct_r - 2xct + x^2 = b^2 + c^2t_r^2;$$

$$2ct_r(x - ct) + (x^2 - 2xct + c^2t^2) = b^2;$$

$$2ct_r(x - ct) = b^2 - (x - ct)^2, \text{ or } t_r = \frac{b^2 - (x - ct)^2}{2c(x - ct)}.$$

Now $V(x,t) = \dfrac{1}{4\pi\epsilon_0}\dfrac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})}$, and $\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \mathscr{r}(c - v)$; $\mathscr{r} = c(t - t_r)$.

$$v = \frac{1}{2}\frac{1}{\sqrt{b^2 + c^2t_r^2}}2c^2t_r = \frac{c^2t_r}{c(t_r - t) + x} = \frac{c^2t_r}{ct_r + (x - ct)};\ (c - v) = \frac{c^2t_r + c(x - ct) - c^2t_r}{ct_r + (x - ct)} = \frac{c(x - ct)}{ct_r + (x - ct)};$$

$$\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v} = \frac{c(t - t_r)c(x - ct)}{ct_r + (x - ct)} = \frac{c^2(t - t_r)(x - ct)}{ct_r + (x - ct)};\ ct_r + (x - ct) = \frac{b^2 - (x - ct)^2}{2(x - ct)} + (x - ct) = \frac{b^2 + (x - ct)^2}{2(x - ct)};$$

$$t - t_r = \frac{2ct(x - ct) - b^2 + (x - ct)^2}{2c(x - ct)} = \frac{(x - ct)(x + ct) - b^2}{2c(x - ct)} = \frac{(x^2 - c^2t^2 - b^2)}{2c(x - ct)}.$$ Therefore

$$\frac{1}{\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}} = \left[\frac{b^2 + (x - ct)^2}{2(x - ct)}\right]\frac{1}{c^2(x - ct)}\frac{2c(x - ct)}{[2ct(x - ct) - b^2 + (x - ct)^2]} = \frac{b^2 + (x - ct)^2}{c(x - ct)\,[2ct(x - ct) - b^2 + (x - ct)^2]}.$$

The term in square brackets simplifies to $(2ct + x - ct)(x - ct) - b^2 = (x + ct)(x - ct) - b^2 = x^2 - c^2t^2 - b^2$.

So $\boxed{V(x,t) = \dfrac{q}{4\pi\epsilon_0}\dfrac{b^2 + (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}.}$

Meanwhile

$$\begin{aligned}\mathbf{A} &= \frac{V}{c^2}\mathbf{v} = \frac{c^2t_r}{ct_r + (x - ct)}\frac{V}{c^2}\,\hat{\mathbf{x}} = \left[\frac{b^2 - (x - ct)^2}{2c(x - ct)}\right]\frac{2(x - ct)}{b^2 + (x - ct)^2}\frac{q}{4\pi\epsilon_0}\frac{b^2 + (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}\,\hat{\mathbf{x}} \\ &= \boxed{\frac{q}{4\pi\epsilon_0 c}\frac{b^2 - (x - ct)^2}{(x - ct)(x^2 - c^2t^2 - b^2)}\,\hat{\mathbf{x}}.}\end{aligned}$$

Problem 10.17

From Eq. 10.33, $c(t - t_r) = \mathscr{r} \Rightarrow c^2(t - t_r)^2 = \mathscr{r}^2 = \boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}$. Differentiate with respect to t:

$2c^2(t - t_r)\left(1 - \dfrac{\partial t_r}{\partial t}\right) = 2\boldsymbol{\mathscr{r}}\cdot\dfrac{\partial\boldsymbol{\mathscr{r}}}{\partial t}$, or $c\mathscr{r}\left(1 - \dfrac{\partial t_r}{\partial t}\right) = \boldsymbol{\mathscr{r}}\cdot\dfrac{\partial\boldsymbol{\mathscr{r}}}{\partial t}$. Now $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$, so

$\dfrac{\partial\boldsymbol{\mathscr{r}}}{\partial t} = -\dfrac{\partial\mathbf{w}}{\partial t} = -\dfrac{\partial\mathbf{w}}{\partial t_r}\dfrac{\partial t_r}{\partial t} = -\mathbf{v}\dfrac{\partial t_r}{\partial t}$; $c\mathscr{r}\left(1 - \dfrac{\partial t_r}{\partial t}\right) = -\boldsymbol{\mathscr{r}}\cdot\mathbf{v}\dfrac{\partial t_r}{\partial t}$; $\quad c\mathscr{r} = \dfrac{\partial t_r}{\partial t}(c\mathscr{r} - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}) = \dfrac{\partial t_r}{\partial t}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})$ (Eq. 10.64),

and hence $\dfrac{\partial t_r}{\partial t} = \dfrac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}$. qed

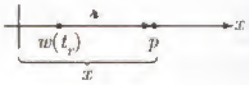
Now Eq. 10.40 says $\mathbf{A}(\mathbf{r},t) = \dfrac{\mathbf{v}}{c^2}V(\mathbf{r},t)$, so

$$\begin{aligned}
\frac{\partial \mathbf{A}}{\partial t} &= \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial t}V + \mathbf{v}\frac{\partial V}{\partial t}\right) = \frac{1}{c^2}\left(\frac{\partial \mathbf{v}}{\partial t_r}\frac{\partial t_r}{\partial t}V + \mathbf{v}\frac{\partial V}{\partial t}\right)\\
&= \frac{1}{c^2}\left[\mathbf{a}\frac{\partial t_r}{\partial t}\frac{1}{4\pi\epsilon_0}\frac{qc}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}} + \mathbf{v}\frac{1}{4\pi\epsilon_0}\frac{-qc}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\frac{\partial}{\partial t}(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})\right]\\
&= \frac{1}{c^2}\frac{qc}{4\pi\epsilon_0}\left[\frac{\mathbf{a}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\frac{\partial t_r}{\partial t} - \frac{\mathbf{v}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left(c\frac{\partial \mathscr{r}}{\partial t} - \frac{\partial \boldsymbol{\mathscr{r}}}{\partial t}\cdot\mathbf{v} - \boldsymbol{\mathscr{r}}\cdot\frac{\partial \mathbf{v}}{\partial t}\right)\right].
\end{aligned}$$

But $\mathscr{r} = c(t-t_r) \Rightarrow \dfrac{\partial \mathscr{r}}{\partial t} = c\left(1 - \dfrac{\partial t_r}{\partial t}\right)$, $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \Rightarrow \dfrac{\partial \boldsymbol{\mathscr{r}}}{\partial t} = -\mathbf{v}\dfrac{\partial t_r}{\partial t}$ (as above), and $\dfrac{\partial \mathbf{v}}{\partial t} = \dfrac{\partial \mathbf{v}}{\partial t_r}\dfrac{\partial t_r}{\partial t} = \mathbf{a}\dfrac{\partial t_r}{\partial t}$.

$$\begin{aligned}
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{\mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\frac{\partial t_r}{\partial t} - \mathbf{v}\left[c^2\left(1-\frac{\partial t_r}{\partial t}\right) + v^2\frac{\partial t_r}{\partial t} - \boldsymbol{\mathscr{r}}\cdot\mathbf{a}\frac{\partial t_r}{\partial t}\right]\right\}\\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v} + \left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{\partial t_r}{\partial t}\right\}\\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^2}\left\{-c^2\mathbf{v} + \left[(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + (c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\frac{c\mathscr{r}}{\boldsymbol{\mathscr{r}}\cdot\mathbf{u}}\right\}\\
&= \frac{q}{4\pi\epsilon_0 c(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[-c^2\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u}) + c\mathscr{r}(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a} + c\mathscr{r}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]\\
&= \frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v})\left(-\mathbf{v} + \frac{\mathscr{r}}{c}\mathbf{a}\right) + \frac{\mathscr{r}}{c}(c^2 - v^2 + \boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right]. \quad \text{qed}
\end{aligned}$$

Problem 10.18



$\mathbf{E} = \dfrac{q}{4\pi\epsilon_0}\dfrac{\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\left[(c^2-v^2)\mathbf{u} + \boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})\right]$. Here $\mathbf{v} = v\,\hat{\mathbf{x}}$, $\mathbf{a} = a\,\hat{\mathbf{x}}$, and, for points to the *right*, $\hat{\boldsymbol{\mathscr{r}}} = \hat{\mathbf{x}}$. So $\mathbf{u} = (c-v)\,\hat{\mathbf{x}}$, $\mathbf{u}\times\mathbf{a} = 0$, and $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c-v)$.

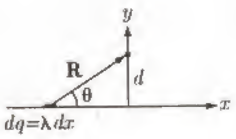
$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c-v)^3}(c^2-v^2)(c-v)\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\frac{(c+v)(c-v)^2}{(c-v)^3}\,\hat{\mathbf{x}} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}};\\
\mathbf{B} &= \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\mathbf{E} = 0. \quad \text{qed}
\end{aligned}$$

For field points to the *left*, $\hat{\boldsymbol{\mathscr{r}}} = -\hat{\mathbf{x}}$ and $\mathbf{u} = -(c+v)\,\hat{\mathbf{x}}$, so $\boldsymbol{\mathscr{r}}\cdot\mathbf{u} = \mathscr{r}(c+v)$, and

$$\mathbf{E} = -\frac{q}{4\pi\epsilon_0}\frac{\mathscr{r}}{\mathscr{r}^3(c+v)^3}(c^2-v^2)(c+v)\,\hat{\mathbf{x}} = \boxed{\frac{-q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}};\ \mathbf{B} = 0.}$$

Problem 10.19

(a) $\mathbf{E} = \dfrac{\lambda}{4\pi\epsilon_0}(1 - v^2/c^2)\displaystyle\int\frac{\hat{\mathbf{R}}}{R^2}\frac{dx}{\left[1-(v/c)^2\sin^2\theta\right]^{3/2}}$.



The horizontal components cancel; the vertical component of $\hat{\mathbf{R}}$ is $\sin\theta$ (see diagram). Here $d = R\sin\theta$, so $\dfrac{1}{R^2} = \dfrac{\sin^2\theta}{d^2}$; $-\dfrac{x}{d} = \cot\theta$, so $dx = -d(-\csc^2\theta)\,d\theta = \dfrac{d}{\sin^2\theta}\,d\theta$;

$\frac{1}{R^2}\,dx = \frac{d}{\sin^2\theta}\frac{\sin^2\theta}{d^2}\,d\theta = \frac{d\theta}{d}$. Thus

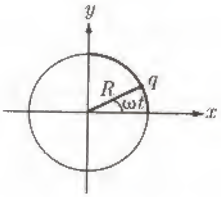
$$\begin{aligned}
\mathbf{E} &= \frac{\lambda}{4\pi\epsilon_0}(1-v^2/c^2)\left(\frac{\hat{\mathbf{y}}}{d}\right)\int_0^\pi \frac{\sin\theta}{\left[1-(v/c)^2\sin^2\theta\right]^{3/2}}\,d\theta. \quad \text{Let } z\equiv\cos\theta, \text{ so } \sin^2\theta = 1-z^2.\\
&= \frac{\lambda(1-v^2/c^2)\,\hat{\mathbf{y}}}{4\pi\epsilon_0 d}\int_{-1}^{1}\frac{1}{[1-(v/c)^2+(v/c)^2z^2]^{3/2}}\,dz\\
&= \frac{\lambda(1-v^2/c^2)\,\hat{\mathbf{y}}}{4\pi\epsilon_0 d}\left.\left[\frac{1}{(v/c)^3}\frac{z}{(c^2/v^2-1)\sqrt{(c/v)^2-1+z^2}}\right]\right|_{-1}^{+1}\\
&= \frac{\lambda(1-v^2/c^2)}{4\pi\epsilon_0 d}\frac{c}{v}\frac{1}{(1-c^2/v^2)}\frac{2}{\sqrt{(c/v)^2-1+1}}\,\hat{\mathbf{y}} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}\,\hat{\mathbf{y}}} \quad \text{(same as for a line charge at rest).}
\end{aligned}$$

(b) $\mathbf{B} = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E})$ for each segment $dq = \lambda\,dx$. Since $\mathbf{v}$ is constant, it comes outside the integral, and the same formula holds for the *total* field:

$$\mathbf{B} = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}) = \frac{1}{c^2}v\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}(\hat{\mathbf{x}}\times\hat{\mathbf{y}}) = \mu_0\epsilon_0 v\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}\,\hat{\mathbf{z}} = \frac{\mu_0}{4\pi}\frac{2\lambda v}{d}\,\hat{\mathbf{z}}.$$

But $\lambda v = I$, so $\boxed{\mathbf{B} = \frac{\mu_0}{4\pi}\frac{2I}{d}\,\hat{\boldsymbol{\phi}}}$ (the same as we got in magnetostatics, Eq. 5.36 and Ex. 5.7).

Problem 10.20



$\mathbf{w}(t) = R[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}]$;
$\mathbf{v}(t) = R\omega[-\sin(\omega t)\,\hat{\mathbf{x}} + \cos(\omega t)\,\hat{\mathbf{y}}]$;
$\mathbf{a}(t) = -R\omega^2[\cos(\omega t)\,\hat{\mathbf{x}} + \sin(\omega t)\,\hat{\mathbf{y}}] = -\omega^2\mathbf{w}(t)$;
$\boldsymbol{\mathscr{r}} = -\mathbf{w}(t_r)$;
$\mathscr{r} = R$;
$t_r = t - R/c$;
$\hat{\boldsymbol{\mathscr{r}}} = -[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}]$;

$$\begin{aligned}
\mathbf{u} &= c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}(t_r) = -c[\cos(\omega t_r)\,\hat{\mathbf{x}} + \sin(\omega t_r)\,\hat{\mathbf{y}}] - \omega R[-\sin(\omega t_r)\,\hat{\mathbf{x}} + \cos(\omega t_r)\,\hat{\mathbf{y}}]\\
&= -\left\{[c\cos(\omega t_r) - \omega R\sin(\omega t_r)]\,\hat{\mathbf{x}} + [c\sin(\omega t_r) + \omega R\cos(\omega t_r)]\,\hat{\mathbf{y}}\right\};\\
\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a}) &= (\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{u} - (\boldsymbol{\mathscr{r}}\cdot\mathbf{u})\mathbf{a};\ \boldsymbol{\mathscr{r}}\cdot\mathbf{a} = -\mathbf{w}\cdot(-\omega^2\mathbf{w}) = \omega^2R^2;\\
\boldsymbol{\mathscr{r}}\cdot\mathbf{u} &= R\left[c\cos^2(\omega t_r) - \omega R\sin(\omega t_r)\cos(\omega t_r) + c\sin^2(\omega t_r) + \omega R\sin(\omega t_r)\cos(\omega t_r)\right] = Rc;
\end{aligned}$$

$v^2 = (\omega R)^2$. So (Eq. 10.65):

$$\begin{aligned}
\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{R}{(Rc)^3}\left[\mathbf{u}\left(c^2-\omega^2R^2\right) + \mathbf{u}(\omega R)^2 - \mathbf{a}(Rc)\right] = \frac{q}{4\pi\epsilon_0}\frac{c\mathbf{u} - R\mathbf{a}}{(Rc)^2}\\
&= \frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\left\{-[c^2\cos(\omega t_r) - \omega Rc\sin(\omega t_r)]\,\hat{\mathbf{x}} - [c^2\sin(\omega t_r) + \omega Rc\cos(\omega t_r)]\,\hat{\mathbf{y}}\right.\\
&\quad \left. + R^2\omega^2\cos(\omega t_r)\,\hat{\mathbf{x}} + R^2\omega^2\sin(\omega t_r)\,\hat{\mathbf{y}}\right\}\\
&= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\left\{\left[\left(\omega^2R^2-c^2\right)\cos(\omega t_r) + \omega Rc\sin(\omega t_r)\right]\hat{\mathbf{x}} + \left[\left(\omega^2R^2-c^2\right)\sin(\omega t_r) - \omega Rc\cos(\omega t_r)\right]\hat{\mathbf{y}}\right\}.}
\end{aligned}$$

$$
\begin{aligned}
\mathbf{B} &= \frac{1}{c}\hat{\mathscr{r}} \times \mathbf{E} = \frac{1}{c}\left(\hat{\mathscr{r}}_x E_y - \hat{\mathscr{r}}_y E_x\right)\hat{\mathbf{z}} \\
&= -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(Rc)^2}\left\{\cos(\omega t_r)\left[\left(\omega^2R^2 - c^2\right)\sin(\omega t_r) - \omega Rc\cos(\omega t_r)\right]\right. \\
&\quad \left. - \sin(\omega t_r)\left[\left(\omega^2R^2 - c^2\right)\cos(\omega t_r) + \omega Rc\sin(\omega t_r)\right]\right\}\hat{\mathbf{z}} \\
&= -\frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\left[-\omega Rc\cos^2(\omega t_r) - \omega Rc\sin^2(\omega t_r)\right]\hat{\mathbf{z}} = \frac{q}{4\pi\epsilon_0}\frac{1}{R^2c^3}\omega Rc\,\hat{\mathbf{z}} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{\omega}{Rc^2}\,\hat{\mathbf{z}}.}
\end{aligned}
$$

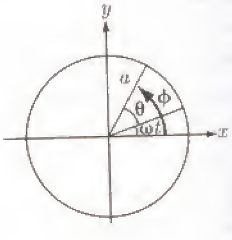
Notice that $\mathbf{B}$ is constant in time.

To obtain the field at the center of a circular *ring* of charge, let $q \to \lambda(2\pi R)$; for this ring to carry current I, we need $I = \lambda v = \lambda\omega R$, so $\lambda = I/\omega R$, and hence $q \to (I/\omega R)(2\pi R) = 2\pi I/\omega$. Thus $\mathbf{B} = \frac{2\pi I}{4\pi\epsilon_0}\frac{1}{Rc^2}\,\hat{\mathbf{z}}$, or, since $1/c^2 = \epsilon_0\mu_0$, $\boxed{\mathbf{B} = \frac{\mu_0 I}{2R}\,\hat{\mathbf{z}},}$ the same as Eq. 5.38, in the case $z = 0$.

Problem 10.21

$\lambda(\phi, t) = \lambda_0|\sin(\theta/2)|$, where $\theta = \phi - \omega t$. So the (retarded) scalar potential at the center is (Eq. 10.19)

$$
\begin{aligned}
V(t) &= \frac{1}{4\pi\epsilon_0}\int\frac{\lambda}{\mathscr{r}}\,dl' = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda_0\left|\sin[(\phi - \omega t_r)/2]\right|}{a}\,a\,d\phi \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\int_0^{2\pi}\sin(\theta/2)\,d\theta = \frac{\lambda_0}{4\pi\epsilon_0}\left[-2\cos(\theta/2)\right]\Big|_0^{2\pi} \\
&= \frac{\lambda_0}{4\pi\epsilon_0}\left[2 - (-2)\right] = \boxed{\frac{\lambda_0}{\pi\epsilon_0}.}
\end{aligned}
$$



(Note: at fixed t_r, $d\phi = d\theta$, and it goes through one full cycle of ϕ *or* θ.)

Meanwhile $\mathbf{I}(\phi, t) = \lambda\mathbf{v} = \lambda_0\omega a\left|\sin[(\phi - \omega t)/2]\right|\hat{\boldsymbol{\phi}}$. From Eq. 10.19 (again)

$$
\mathbf{A}(t) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{I}}{\mathscr{r}}\,dl' = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{\lambda_0\omega a\left|\sin[(\phi - \omega t_r)/2]\right|\hat{\boldsymbol{\phi}}}{a}\,a\,d\phi.
$$

But $t_r = t - a/c$ is again constant, for the ϕ integration, and $\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$.

$$
= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}\left|\sin[(\phi - \omega t_r)/2]\right|\left(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}\right)d\phi.
$$

Again, switch variables to $\theta = \phi - \omega t_r$, and integrate from $\theta = 0$ to $\theta = 2\pi$ (so we don't have to worry about the absolute value).

$$
= \frac{\mu_0\lambda_0\omega a}{4\pi}\int_0^{2\pi}\sin(\theta/2)\left[-\sin(\theta + \omega t_r)\,\hat{\mathbf{x}} + \cos(\theta + \omega t_r)\,\hat{\mathbf{y}}\right]d\theta.
$$

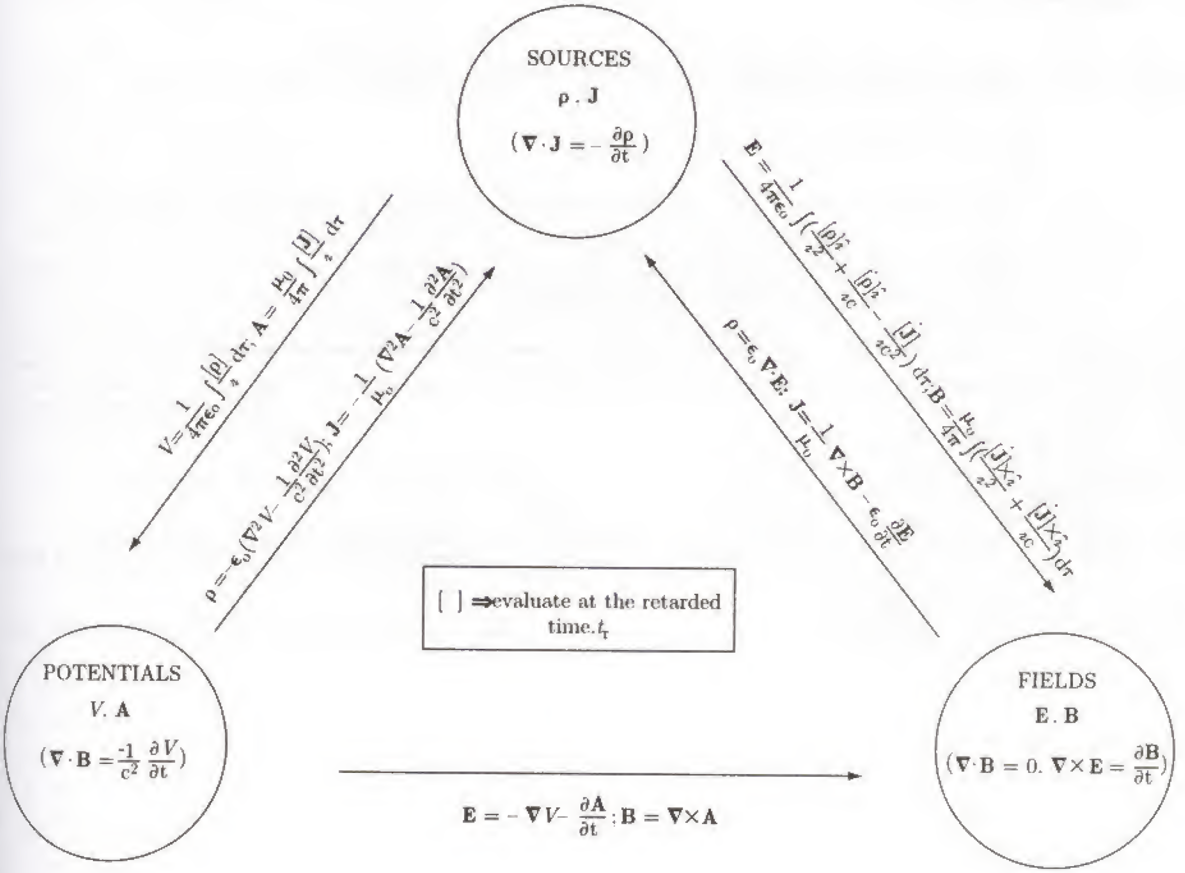
Now

$$\begin{aligned}
\int_0^{2\pi} \sin(\theta/2)\sin(\theta+\omega t_r)\,d\theta &= \frac{1}{2}\int_0^{2\pi}\left[\cos(\theta/2+\omega t_r)-\cos(3\theta/2+\omega t_r)\right]d\theta \\
&= \frac{1}{2}\left[2\sin(\theta/2+\omega t_r)-\frac{2}{3}\sin(3\theta/2+\omega t_r)\right]\Bigg|_0^{2\pi} \\
&= \sin(\pi+\omega t_r)-\sin(\omega t_r)-\frac{1}{3}\sin(3\pi+\omega t_r)+\frac{1}{3}\sin(\omega t_r) \\
&= -2\sin(\omega t_r)+\frac{2}{3}\sin(\omega t_r) = -\frac{4}{3}\sin(\omega t_r). \\
\int_0^{2\pi} \sin(\theta/2)\cos(\theta+\omega t_r)\,d\theta &= \frac{1}{2}\int_0^{2\pi}\left[-\sin(\theta/2+\omega t_r)+\sin(3\theta/2+\omega t_r)\right]d\theta \\
&= \frac{1}{2}\left[2\cos(\theta/2+\omega t_r)-\frac{2}{3}\cos(3\theta/2+\omega t_r)\right]\Bigg|_0^{2\pi} \\
&= \cos(\pi+\omega t_r)-\cos(\omega t_r)-\frac{1}{3}\cos(3\pi+\omega t_r)+\frac{1}{3}\cos(\omega t_r) \\
&= -2\cos(\omega t_r)+\frac{2}{3}\cos(\omega t_r) = -\frac{4}{3}\cos(\omega t_r).
\end{aligned}$$

So

$$\mathbf{A}(t) = \frac{\mu_0\lambda_0\omega a}{4\pi}\left(\frac{4}{3}\right)\left[\sin(\omega t_r)\,\hat{\mathbf{x}}-\cos(\omega t_r)\,\hat{\mathbf{y}}\right] = \boxed{\frac{\mu_0\lambda_0\omega a}{3\pi}\left\{\sin[\omega(t-a/c)]\,\hat{\mathbf{x}}-\cos[\omega(t-a/c)]\,\hat{\mathbf{y}}\right\}.}$$

Problem 10.22


SOURCES
$\rho\,.\,\mathbf{J}$
$(\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t})$
$V=\frac{1}{4\pi\epsilon_0}\int\frac{[\rho]}{\mathscr{r}}d\tau;\ \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{[\mathbf{J}]}{\mathscr{r}}d\tau$
$\rho=-\epsilon_0(\nabla^2V-\frac{1}{c^2}\frac{\partial^2V}{\partial t^2});\ \mathbf{J}=-\frac{1}{\mu_0}(\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2})$
$\mathbf{E}=\frac{1}{4\pi\epsilon_0}\int(\frac{[\rho]\hat{\mathscr{r}}}{\mathscr{r}^2}+\frac{[\dot{\rho}]\hat{\mathscr{r}}}{\mathscr{r}c}-\frac{[\dot{\mathbf{J}}]}{\mathscr{r}c^2})d\tau;\mathbf{B}=\frac{\mu_0}{4\pi}\int(\frac{[\mathbf{J}]\times\hat{\mathscr{r}}}{\mathscr{r}^2}+\frac{[\dot{\mathbf{J}}]\times\hat{\mathscr{r}}}{\mathscr{r}c})d\tau$
$\rho=\epsilon_0\,\nabla\cdot\mathbf{E};\ \mathbf{J}=\frac{1}{\mu_0}\,\nabla\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$
[ ] ⇒evaluate at the retarded time.t_r
POTENTIALS
$V.\ \mathbf{A}$
$(\nabla\cdot\mathbf{B}=\frac{-1}{c^2}\frac{\partial V}{\partial t})$
FIELDS
$\mathbf{E}\,.\,\mathbf{B}$
$(\nabla\cdot\mathbf{B}=0.\ \nabla\times\mathbf{E}=\frac{\partial\mathbf{B}}{\partial t})$
$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t};\mathbf{B}=\nabla\times\mathbf{A}$

Problem 10.23

Using Product Rule #5, Eq. 10.43 ⇒

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= \frac{\mu_0}{4\pi}qc\mathbf{v}\cdot\nabla\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-1/2}\\
&= \frac{\mu_0 qc}{4\pi}\mathbf{v}\cdot\left\{-\frac{1}{2}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\nabla\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\right\}\\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left\{-2(c^2t-\mathbf{r}\cdot\mathbf{v})\nabla(\mathbf{r}\cdot\mathbf{v})+(c^2-v^2)\nabla(r^2)\right\}.
\end{aligned}$$

Product Rule #4 ⇒

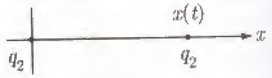
$$\begin{aligned}
\nabla(\mathbf{r}\cdot\mathbf{v}) &= \mathbf{v}\times(\nabla\times\mathbf{r})+(\mathbf{v}\cdot\nabla)\mathbf{r},\ \text{but }\nabla\times\mathbf{r}=0,\\
(\mathbf{v}\cdot\nabla)\mathbf{r} &= \left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}})=v_x\,\hat{\mathbf{x}}+v_y\,\hat{\mathbf{y}}+v_z\,\hat{\mathbf{z}}=\mathbf{v},\ \text{and}\\
\nabla(r^2) &= \nabla(\mathbf{r}\cdot\mathbf{r})=2\mathbf{r}\times(\nabla\times\mathbf{r})+2(\mathbf{r}\cdot\nabla)\mathbf{r}=2\mathbf{r}.\ \text{So}
\end{aligned}$$

$$\begin{aligned}
\nabla\cdot\mathbf{A} &= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\mathbf{v}\cdot\left[-2(c^2t-\mathbf{r}\cdot\mathbf{v})\mathbf{v}+(c^2-v^2)2\mathbf{r}\right]\\
&= \frac{\mu_0 qc}{4\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left\{(c^2t-\mathbf{r}\cdot\mathbf{v})v^2-(c^2-v^2)(\mathbf{r}\cdot\mathbf{v})\right\}.\\
&\ \text{But the term in curly brackets is : } c^2tv^2-v^2(\mathbf{r}\cdot\mathbf{v})-c^2(\mathbf{r}\cdot\mathbf{v})+v^2(\mathbf{r}\cdot\mathbf{v})=c^2(v^2t-\mathbf{r}\cdot\mathbf{v}).\\
&= \frac{\mu_0 qc^3}{4\pi}\frac{(v^2t-\mathbf{r}\cdot\mathbf{v})}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}.
\end{aligned}$$

Meanwhile, from Eq. 10.42,

$$\begin{aligned}
-\mu_0\epsilon_0\frac{\partial V}{\partial t} &= -\mu_0\epsilon_0\frac{1}{4\pi\epsilon_0}qc\left(-\frac{1}{2}\right)\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\ \times\\
&\ \frac{\partial}{\partial t}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]\\
&= -\frac{\mu_0 qc}{8\pi}\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{-3/2}\left[2(c^2t-\mathbf{r}\cdot\mathbf{v})c^2+(c^2-v^2)(-2c^2t)\right]\\
&= -\frac{\mu_0 qc^3}{4\pi}\frac{(c^2t-\mathbf{r}\cdot\mathbf{v}-c^2t+v^2t)}{\left[(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\right]^{3/2}}=\nabla\cdot\mathbf{A}.\ \checkmark
\end{aligned}$$

Problem 10.24



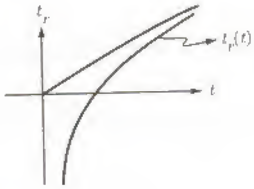
(a) $\boxed{\mathbf{F}_2=\frac{q_1q_2}{4\pi\epsilon_0}\frac{1}{(b^2+c^2t^2)}\,\hat{\mathbf{x}}.}$

(This is just Coulomb's law, since q_1 is at rest.)

(b) $I_2=\frac{q_1q_2}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{1}{(b^2+c^2t^2)}\,dt=\frac{q_1q_2}{4\pi\epsilon_0}\left[\frac{1}{bc}\tan^{-1}(ct/b)\right]\Big|_{-\infty}^{\infty}=\frac{q_1q_2}{4\pi\epsilon_0 bc}\left[\tan^{-1}(\infty)-\tan^{-1}(-\infty)\right]$

$=\frac{q_1q_2}{4\pi\epsilon_0 bc}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=\boxed{\frac{q_1q_2}{4\pi\epsilon_0}\frac{\pi}{bc}.}$

(c) From Prob. 10.18, $\mathbf{E} = -\frac{q_2}{4\pi\epsilon_0}\frac{1}{x^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}}$. Here x and v are to be evaluated at the retarded time t_r, which is given by $c(t-t_r) = x(t_r) = \sqrt{b^2+c^2t_r^2} \Rightarrow c^2t^2 - 2ctt_r + c^2t_r^2 = b^2 + c^2t_r^2 \Rightarrow t_r = \frac{c^2t^2-b^2}{2c^2t}$. *Note:* As we found in Prob. 10.15, q_2 first "comes into view" (for q_1) at time $t=0$. Before that it can exert no force on q_1, and there *is* no retarded time. From the graph of t_r versus t we see that t_r ranges all the way from $-\infty$ to ∞ while $t>0$.



$$x(t_r) = c(t-t_r) = \frac{2c^2t^2 - c^2t^2 + b^2}{2ct} = \frac{b^2+c^2t^2}{2ct} \text{ (for } t>0). \quad v(t) = \frac{1}{2}\frac{2c^2t}{\sqrt{b^2+c^2t^2}} = \frac{c^2t}{x}, \text{ so}$$

$$v(t_r) = \left(\frac{c^2t^2-b^2}{2t}\right)\left(\frac{2ct}{b^2+c^2t^2}\right) = c\left(\frac{c^2t^2-b^2}{c^2t^2+b^2}\right) \text{ (for } t>0). \quad \text{Therefore}$$

$$\frac{c-v}{c+v} = \frac{(c^2t^2+b^2)-(c^2t^2-b^2)}{(c^2t^2+b^2)+(c^2t^2-b^2)} = \frac{2b^2}{2c^2t^2} = \frac{b^2}{c^2t^2} \text{ (for } t>0). \quad \mathbf{E} = -\frac{q_2}{4\pi\epsilon_0}\frac{4c^2t^2}{(b^2+c^2t^2)^2}\frac{b^2}{c^2t^2}\hat{\mathbf{x}} \Rightarrow$$

$$\boxed{\mathbf{F}_1 = \begin{cases} 0, & t<0; \\ -\frac{q_1q_2}{4\pi\epsilon_0}\frac{4b^2}{(b^2+c^2t^2)^2}\hat{\mathbf{x}}, & t>0. \end{cases}}$$

(d) $I_1 = -\frac{q_1q_2}{4\pi\epsilon_0}4b^2\int_0^\infty \frac{1}{(b^2+c^2t^2)^2}\,dt$. The integral is

$$\frac{1}{c^4}\int_0^\infty \frac{1}{[(b/c)^2+t^2]^2}\,dt = \frac{1}{c^4}\left(\frac{c^2}{2b^2}\right)\left[\frac{t}{(b/c)^2+t^2}\Bigg|_0^\infty + \int_0^\infty \frac{1}{[(b/c)^2+t^2)]}\,dt\right] = \frac{1}{2c^2b^2}\left(\frac{\pi c}{2b}\right) = \frac{\pi}{4cb^3}.$$

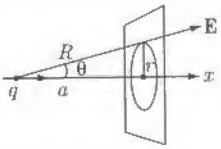
So $\boxed{I_1 = -\frac{q_1q_2}{4\pi\epsilon_0}\frac{\pi}{bc}.}$

(e) $\mathbf{F}_1 \neq -\mathbf{F}_2$, so Newton's third law is *not* obeyed. On the other hand, $I_1 = -I_2$ in this instance, which suggests that the *net* momentum delivered from (1) to (2) is equal and opposite to the net momentum delivered from (2) to (1), and hence that the total mechanical momentum is conserved. (In general, the fields might carry off some momentum, leaving the mechanical momentum altered; but that doesn't happen in the present case.)

Problem 10.25

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}); \; \mathbf{B} = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}) \text{ (Eq. 10.69).}$$

So $\mathbf{S} = \frac{1}{\mu_0c^2}\left[\mathbf{E}\times(\mathbf{v}\times\mathbf{E})\right] = \epsilon_0\left[E^2\mathbf{v} - (\mathbf{v}\cdot\mathbf{E})\mathbf{E}\right]$.



The power crossing the plane is $P = \int \mathbf{S}\cdot d\mathbf{a}$, and $d\mathbf{a} = 2\pi r\,dr\,\hat{\mathbf{x}}$ (see diagram). So

$$\begin{aligned}
P &= \epsilon_0 \int (E^2 v - E_x^2 v) 2\pi r\,dr;\ E_x = E\cos\theta,\ \text{so } E^2 - E_x^2 = E^2\sin^2\theta. \\
&= 2\pi\epsilon_0 v \int E^2 \sin^2\theta\, r\,dr.\ \text{ From Eq. 10.68, } \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\gamma^2}\frac{\hat{\mathbf{R}}}{R^2\left[1-(v/c)^2\sin^2\theta\right]^{3/2}} \text{ where } \gamma \equiv \frac{1}{\sqrt{1-v^2/c^2}}. \\
&= 2\pi\epsilon_0 v \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{1}{\gamma^2}\int_0^\infty \frac{r\sin^2\theta}{R^4\left[1-(v/c)^2\sin^2\theta\right]^3}\,dr.\ \text{ Now } r = a\tan\theta \Rightarrow dr = a\frac{1}{\cos^2\theta}\,d\theta;\ \frac{1}{R} = \frac{\cos\theta}{a}. \\
&= \frac{v}{2\gamma^4}\frac{q^2}{4\pi\epsilon_0}\frac{1}{a^2}\int_0^{\pi/2}\frac{\sin^3\theta\cos\theta}{\left[1-(v/c)^2\sin^2\theta\right]^3}\,d\theta.\ \text{ Let } u \equiv \sin^2\theta,\ \text{so } du = 2\sin\theta\cos\theta\,d\theta. \\
&= \frac{vq^2}{16\pi\epsilon_0 a^2\gamma^4}\int_0^1 \frac{u}{\left[1-(v/c)^2u\right]^3}\,du = \frac{vq^2}{16\pi\epsilon_0 a^2\gamma^4}\left(\frac{\gamma^4}{2}\right) = \boxed{\frac{vq^2}{32\pi\epsilon_0 a^2}.}
\end{aligned}$$

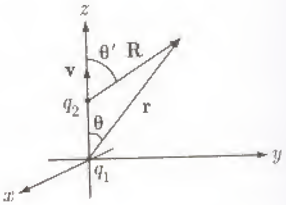
Problem 10.26

(a) $\boxed{\mathbf{F}_{12}(t) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{(vt)^2}\,\hat{\mathbf{z}}.}$

(b) From Eq. 10.68, with $\theta = 180°$, $R = vt$, and $\hat{\mathbf{R}} = -\hat{\mathbf{z}}$:

$$\boxed{\mathbf{F}_{21}(t) = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2(1-v^2/c^2)}{(vt)^2}\,\hat{\mathbf{z}}.}$$

$\boxed{\text{No,}}$ Newton's third law does *not* hold: $\mathbf{F}_{12} \neq \mathbf{F}_{21}$, because of the extra factor $(1 - v^2/c^2)$.



(c) From Eq. 8.29, $\mathbf{p} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$. Here $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, whereas $\mathbf{B} = \mathbf{B}_2$, so $\mathbf{E}\times\mathbf{B} = (\mathbf{E}_1\times\mathbf{B}_2) + (\mathbf{E}_2\times\mathbf{B}_2)$. But the latter, when integrated over all space, is independent of time. We want only the time-dependent part: $\mathbf{p}(t) = \epsilon_0\int(\mathbf{E}_1\times\mathbf{B}_2)\,d\tau$. Now $\mathbf{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\,\hat{\mathbf{r}}$, while, from Eq. 10.69, $\mathbf{B}_2 = \frac{1}{c^2}(\mathbf{v}\times\mathbf{E}_2)$, and (Eq. 10.68) $\mathbf{E}_2 = \frac{q_2}{4\pi\epsilon_0}\frac{(1-v^2/c^2)}{(1-v^2\sin^2\theta'/c^2)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}$. But $\mathbf{R} = \mathbf{r} - \mathbf{v}t$; $R^2 = r^2 + v^2t^2 - 2rvt\cos\theta$; $\sin\theta' = \frac{r\sin\theta}{R}$. So $\mathbf{E}_2 = \frac{q_2}{4\pi\epsilon_0}\frac{(1-v^2/c^2)}{\left[1-(vr\sin\theta/Rc)^2\right]^{3/2}}\frac{(\mathbf{r}-\mathbf{v}t)}{R^3}$. Finally, noting that $\mathbf{v}\times(\mathbf{r}-\mathbf{v}t) = \mathbf{v}\times\mathbf{r} = vr\sin\theta\,\hat{\boldsymbol{\phi}}$, we get $\mathbf{B}_2 = \frac{q_2(1-v^2/c^2)}{4\pi\epsilon_0 c^2}\frac{vr\sin\theta}{\left[R^2-(vr\sin\theta/c)^2\right]^{3/2}}\,\hat{\boldsymbol{\phi}}$. So $\mathbf{p}(t) = \epsilon_0\frac{q_1}{4\pi\epsilon_0}\frac{q_2(1-v^2/c^2)v}{4\pi\epsilon_0 c^2}\int\frac{1}{r^2}\frac{r\sin\theta\,(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}})}{\left[R^2-(vr\sin\theta/c)^2\right]^{3/2}}$. But $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}} = -(\cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}})$, and the x and y components integrate to zero, so:

$$\begin{aligned}
\mathbf{p}(t) &= \frac{q_1q_2v(1-v^2/c^2)\,\hat{\mathbf{z}}}{(4\pi c)^2\epsilon_0}\int\frac{\sin^2\theta}{r\left[r^2+(vt)^2-2rvt\cos\theta-(vr\sin\theta/c)^2\right]^{3/2}}\,r^2\sin\theta\,dr\,d\theta\,d\phi \\
&= \frac{q_1q_2v(1-v^2/c^2)\,\hat{\mathbf{z}}}{8\pi c^2\epsilon_0}\int\frac{r\sin^3\theta}{\left[r^2+(vt)^2-2rvt\cos\theta-(vr\sin\theta/c)^2\right]^{3/2}}\,dr\,d\theta.
\end{aligned}$$

I'll do the r integral first. According to the CRC Tables,

$$\begin{aligned}\int_0^\infty \frac{x}{(a+bx+cx^2)^{3/2}}\,dx &= -\frac{2(bx+2a)}{(4ac-b^2)\sqrt{a+bx+cx^2}}\bigg|_0^\infty = -\frac{2}{4ac-b^2}\left[\frac{b}{\sqrt{c}}-\frac{2a}{\sqrt{a}}\right]\\ &= -\frac{2}{\sqrt{c}(4ac-b^2)}\left(b-2\sqrt{ac}\right) = \frac{2}{\sqrt{c}}\frac{(2\sqrt{ac}-b)}{(2\sqrt{ac}-b)\,(2\sqrt{ac}+b)} = \frac{2}{\sqrt{c}}\left(2\sqrt{ac}+b\right).\end{aligned}$$

In this case $x=r$, $a=(vt)^2$, $b=-2vt\cos\theta$, and $c=1-(v/c)^2\sin^2\theta$. So the r integral is

$$\frac{2}{\sqrt{1-(v/c)^2\sin^2\theta}\left[2vt\sqrt{1-(v/c)^2\sin^2\theta}-2vt\cos\theta\right]} = \frac{1}{vt\sqrt{1-(v/c)^2\sin^2\theta}\left[\sqrt{1-(v/c)^2\sin^2\theta}-\cos\theta\right]}$$

$$= \frac{\left[\sqrt{1-(v/c)^2\sin^2\theta}+\cos\theta\right]}{vt\sqrt{1-(v/c)^2\sin^2\theta}\left[1-(v/c)^2\sin^2\theta-\cos^2\theta\right]} = \frac{1}{vt\sin^2\theta(1-v^2/c^2)}\left[1+\frac{\cos\theta}{\sqrt{1-(v/c)^2\sin^2\theta}}\right].$$

So

$$\begin{aligned}\mathbf{p}(t) &= \frac{q_1q_2v(1-v^2/c^2)\,\hat{\mathbf{z}}}{8\pi c^2\epsilon_0}\frac{1}{vt(1-v^2/c^2)}\int_0^\pi \frac{1}{\sin^2\theta}\left[1+\frac{\cos\theta}{\sqrt{1-(v/c)^2\sin^2\theta}}\right]\sin^3\theta\,d\theta\\ &= \frac{q_1q_2\,\hat{\mathbf{z}}}{8\pi c^2\epsilon_0 t}\left\{\int_0^\pi \sin\theta\,d\theta + \frac{c}{v}\int_0^\pi \frac{\cos\theta\sin\theta}{\sqrt{(c/v)^2-\sin^2\theta}}\,d\theta\right\}.\end{aligned}$$

But $\int_0^\pi \sin\theta\,d\theta = 2$. In the second integral let $u\equiv\cos\theta$, so $du=-\sin\theta\,d\theta$:

$\displaystyle\int_0^\pi \frac{\cos\theta\sin\theta}{\sqrt{(c/v)^2-\sin^2\theta}}\,d\theta = \int_{-1}^1 \frac{u}{\sqrt{(c/v)^2-1+u^2}}\,du = 0$ (the integrand is odd, and the interval is even).

Conclusion: $\boxed{\mathbf{p}(t) = \dfrac{\mu_0q_1q_2}{4\pi t}\,\hat{\mathbf{z}}}$ (plus a term constant in time).

(d)

$$\begin{aligned}\mathbf{F}_{12}+\mathbf{F}_{21} &= \frac{1}{4\pi\epsilon_0}\frac{q_1q_2}{v^2t^2}\,\hat{\mathbf{z}} - \frac{1}{4\pi\epsilon_0}\frac{q_1q_2(1-v^2/c^2)}{v^2t^2}\,\hat{\mathbf{z}} = \frac{q_1q_2}{4\pi\epsilon_0v^2t^2}\left(1-1+\frac{v^2}{c^2}\right)\hat{\mathbf{z}} = \frac{q_1q_2}{4\pi\epsilon_0c^2t^2}\,\hat{\mathbf{z}} = \frac{\mu_0q_1q_2}{4\pi t^2}\,\hat{\mathbf{z}}.\\ -\frac{d\mathbf{p}}{dt} &= \frac{\mu_0q_1q_2}{4\pi t^2}\,\hat{\mathbf{z}} = \mathbf{F}_{12}+\mathbf{F}_{21}.\quad \text{qed}\end{aligned}$$

Since q_1 is at rest, and q_2 is moving at constant velocity, there must be another force ($\mathbf{F}_{\text{mech}}$) acting on them, to balance $\mathbf{F}_{12}+\mathbf{F}_{21}$; what we have found is that $\mathbf{F}_{\text{mech}} = d\mathbf{p}_{\text{em}}/dt$, which means that the impulse imparted to the system by the external force ends up as momentum in the fields. [For further discussion of this problem see J. J. G. Scanio, *Am. J. Phys.* **43**, 258 (1975).]